Lecture 2

1. Appendix

1) Proof of Ascoli-Arzela Lemma

Step 1. Construction of the desired subsequence.

Take the whole rational numbers $\{r_i\}$ in $[a,b]$. Since $\{f(r_1)\}$ is bounded in R^n, there exists a convergent subsequence $\{f_{1k}(r_1)\}$ $(k \in N^+)$ by Bolzano-Weierstrass Theorem, i.e. $\{f_{1k}(t)\}$ converges at $t = r_1$. For $\{f_{1k}(r_2)\}$, which is still bounded in R^n, there exists a convergent subsequence $\{f_{2k}(r_2)\} \subset \{f_{1k}(r_2)\}$ by the same reason. $\{f_{2k}(t)\}$ converges at $t = r_1, r_2$. After similar n steps, we can find following countable subsequences:

$$\begin{array}{ccccc} f_{11}(t) & f_{12}(t) & \cdots & f_{1n}(t) & \cdots \\ f_{21}(t) & f_{22}(t) & \cdots & f_{2n}(t) & \cdots \\ \vdots & \vdots & \ddots & \vdots & \cdots, \\ f_{n1}(t) & f_{n2}(t) & \cdots & f_{nn}(t) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{array}$$

where $\{f_{nk}(t)\}$ converges at $t = r_1, r_2, \cdots, r_n$.

Taking the diagonal sequence $g_n(t) = f_{nn}(t)$ $(n \in N^+)$, this sequence converges at any $\{r_i\}$ by the construction because $\{g_n(r_i)\}_{n \ge i} = \{f_{nn}(r_i)\}_{n \ge i}$ is a subsequence of $\{f_{in}(r_i)\}_{n \ge i}$ which converges.

Step 2. It remains to show that $g_n(t)$ is uniformly convergent on $[a,b]$, i.e. $\forall\, \varepsilon > 0$, there exists $N \ge 1$ s.t. $n, m \ge N \;\Rightarrow\; \| g_n(t) - g_m(t) \| < \varepsilon$.

Since $g_n(t)$ converges on $\{r_i\}$, we have that for $\forall\, \varepsilon > 0$ and any $r_i \in [a,b]$, there exists $N(r_i)$ s.t. $n,\ m > N(r_i) \;\Rightarrow\; \| g_n(r_i) - g_m(r_i) \| < \dfrac{\varepsilon}{3}$.

By equicontinuity, and for the above given $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ s.t. for any $t_1, t_2 \in [a,b]$, $|t_1 - t_2| < \delta(\varepsilon) \;\Rightarrow\; \| g_n(t_1) - g_n(t_2) \| < \dfrac{\varepsilon}{3}$ for all $n \in N^+$.

Taking $O(r_i, \delta(\varepsilon))$, then $\bigcup_{i=1}^{\infty} O(r_i, \delta(\varepsilon)) \supseteq [a,b]$. There exists a finite numbers of $O(r_i, \delta(\varepsilon))$ $(i=1, \quad , p)$ s.t. $\bigcup_{i=1}^{p} O(r_i, \delta(\varepsilon)) \supseteq [a,b]$.

Let $N = \max\{N(r_1), N(r_2), \cdots, N(r_p)\}$. Once $n, m \ge N$ and $t \in [a,b]$, there exists one $O(r_{i_0}, \delta(\varepsilon))$, $1 \le i_0 \le p$ s.t. $t \in O(r_{i_0}, \delta(\varepsilon))$. Then

$$\| g_n(t) - g_m(t) \| \le \| g_n(t) - g_n(r_{i_0}) \| + \| g_n(r_{i_0}) - g_m(r_{i_0}) \| + \| g_m(r_{i_0}) - g_m(t) \|$$
$$\le \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon .$$

This completes the proof. □

2) Proof of Peano Theorem by a Traditional Way

Construction for $t \in I_+$ only (the construction for $t \in I_-$ is similar).

Step 1. Construction:

For each $m \ge 1$, we subdivide I_+ with $I_+ = \bigcup_{k=1}^{m} [t_{k-1}^{(m)}, t_k^{(m)}]$, where $t_k^{(m)} = t_0 + \frac{hk}{m}$ for $k = 1, 2, \quad , m$. We define $x_m(t)$ step by step on each subinterval $[t_{k-1}^{(m)}, t_k^{(m)}]$:

$$x_m(t) = x_0 + f(t_0, x_0)(t - t_0) \quad \text{for} \quad t \in [t_0^{(m)}, t_1^{(m)}], \text{ where } t_0^{(m)} = t_0 ,$$
$$\Rightarrow \quad x_m(t_1^{(m)}) = x_0 + f(t_0, x_0)(t_1^{(m)} - t_0) ;$$
$$x_m(t) = x_m(t_1^{(m)}) + f(t_1^{(m)}, x_m(t_1^{(m)}))(t - t_1^{(m)}) \quad \text{for} \quad t \in [t_1^{(m)}, t_2^{(m)}],$$
$$\Rightarrow \quad x_m(t_2^{(m)}) = x_m(t_1^{(m)}) + f(t_1^{(m)}, x_m(t_1^{(m)}))(t_2^{(m)} - t_1^{(m)}) ;$$

By induction, if we have constructed $x_m(t) = x_m(t_{k-1}^{(m)}) + f(t_{k-1}^{(m)}, x_m(t_{k-1}^{(m)}))(t - t_{k-1}^{(m)})$ for $t \in [t_{k-1}^{(m)}, t_k^{(m)}]$, so we have $x_m(t_k^{(m)}) = x_m(t_{k-1}^{(m)}) + f(t_{k-1}^{(m)}, x_m(t_{k-1}^{(m)}))(t_k^{(m)} - t_{k-1}^{(m)})$. Then, we define

$$x_m(t) = x_m(t_k^{(m)}) + f(t_k^{(m)}, x_m(t_k^{(m)}))(t - t_k^{(m)}) \quad \text{for} \quad t \in [t_k^{(m)}, t_{k+1}^{(m)}] .$$

So we have defined $x_m(t)$ on all $t \in I_+$, which is called the Euler polygons.

Step 2. $\{x_m(t)\}$ is well defined on I_+:

Since

$$\|x_m(t)-x_0\| \le \|x_m(t)-x_m(t_k^{(m)})\|+\|x_m(t_k^{(m)})-x_m(t_{k-1}^{(m)})\|+\cdots+\|x_m(t_1^{(m)})-x_0\|$$

$$= \|f(t_k^{(m)},x_m(t_k^{(m)}))(t-t_k^{(m)})\|+\|f(t_{k-1}^{(m)},x_m(t_{k-1}^{(m)}))(t_{k-1}^{(m)}-t_k^{(m)})\|$$

$$+\cdots+\|f(t_0,x_0)(t_1^{(m)}-t_0)\|$$

$$\le M(t-t_k^{(m)})+M(t_{k-1}^{(m)}-t_k^{(m)})+\cdots+M(t_1^{(m)}-t_0)$$

$$= M(t-t_0)\le Mh\le b,\quad t_0\le t\le t_{k+1}^{(m)}.$$

So that $(t,\, x_m(t))\in Q$ for $t_0\le t\le t_{k+1}^{(m)}$ $\Rightarrow$ $\{x_m(t)\}$ is well defined on I_+.

Step 3. Since $x_m(t)$ is continuous at $t_k^{(m)}$, and $x_m(t)$ has a derivative $f(t_k^{(m)},x_m(t_k^{(m)}))$ on $[t_k^{(m)},\, t_{k+1}^{(m)}]$, then,

$$x_m(t)=x_0+\int_{t_0}^{t} f^{(m)}(s)ds \quad \text{for} \quad t_0\le t\le t_{k+1}^{(m)},$$

where $f^{(m)}(t):=f(t_j^{(m)},x_m(t_j^{(m)}))$ for $t\in[t_j^{(m)},\, t_{j+1}^{(m)}]$ (piecewise function).

Step 4. Show that $\{x_m(t)\}$ is equicontinuous and uniformly bounded.

For $t',\, t\in I_+$,

$$\|x_m(t)-x_m(t')\|=\|\int_t^{t'} f^{(m)}(s)ds\|\le M\,|t-t'|,\quad \forall\, m\ge 1$$

and

$$\|x_m(t)\|\le\|x_m(t_0)\|+\|x_m(t)-x_m(t_0)\|\le\|x_m(t_0)\|+Mh,$$

hence, $\{x_m(t)\}$ is equicontinuous and uniformly bounded.

Step 5. Applying Ascoli–Arzela lemma, we have $x_{m_j}(t)\overset{E}{\rightarrow} x(t)$ on I_+, where $\{x_{m_j}(t)\}\subseteq\{x_m(t)\}$. We claim that $x(t)$ is the desired solution of (E) if $f^{(m_j)}(t)\overset{E}{\rightarrow} f(t,\, x(t))$ on I_+.

Step 6. Show that $f^{(m_j)}(t) \overset{E}{\to} f(t, x(t))$ on I_+:

For simple notation, we suppose that $x_m(t) \overset{E}{\to} x(t)$ on I_+. Then, for the given $\varepsilon > 0$, since $f \in C(Q)$ and Q is compact, $\exists\, \delta > 0$ such that

$$\| f(t, x) - f(t', x') \| < \varepsilon$$

whenever $\| (t - t', x - x')^T \| < \delta$, (uniformly continuous).

Now choose m so large such that $\dfrac{h}{m} < \dfrac{\delta}{3}$, $\dfrac{Mh}{m} < \dfrac{\delta}{3}$ and $\| x_m(t) - x(t) \| < \dfrac{\delta}{3}$ whenever $t \in I_+$. For $t \in I_+$, $t \in [t_k^{(m)}, t_{k+1}^{(m)}]$ for some k, we have

$$\begin{aligned}
\| (t_k^{(m)} - t, x_m(t_k^{(m)}) - x(t))^T \| &\le | t_k^{(m)} - t | + \| x_m(t_k^{(m)}) - x(t)) \| \\
&\le | t_k^{(m)} - t | + \| x_m(t_k^{(m)}) - x_m(t) \| + \| x_m(t) - x(t) \| \\
&\le \frac{h}{m} + \frac{Mh}{m} + \frac{\delta}{3} \le \frac{\delta}{3} + \frac{\delta}{3} + \frac{\delta}{3} = \delta ,
\end{aligned}$$

and hence

$$\| f^{(m)}(t) - f(t, x(t)) \| = \| f^{(m)}(t_k^{(m)}, x_m(t_k^{(m)})) - f(t, x(t)) \| < \varepsilon . \quad \square$$